Math 240 **Name:** ______________________
Spring 2020
Practice 1
2/20/2020
Time Limit: 80 Minutes ID ______________________

"My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this"

Signature ______________________

This exam contains 11 pages (including this cover page) and 8 questions.
Total of points is 80.

- Check your exam to make sure all 11 pages are present.
- You may use writing implements on both sides of a sheet of 8"x11" paper.
- NO CALCULATORS.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

1. (10 points) Suppose A is a 3×3 matrix who has

$$\text{Ker}(A) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ such that } a + b + c = 0 \right\}$$

What is the reduced row echelon form of A?

$$\text{rref}(A) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2. (10 points) Let $V = \{y = f(x) \mid y'' + xy^c = 0\}$ where c is some fixed positive integer. For which value of c is V a vector space?

$c = 1$.

① closed under addition:

$y_1, y_2 \in V$.

$y_1'' + xy_1 = 0$, $y_2'' + xy_2 = 0$

$(y_1 + y_2)'' + x(y_1 + y_2)$

$= (y_1'' + xy_1) + (y_2'' + xy_2) = 0 + 0$

$= 0$

② closed under addition

$(cy)'' + x(cy) = c(y'' + xy) = 0$

if $y \in V$.

3. (10 points) Let

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}, \quad w_1 = \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$$

Geometrically, what are span(v_1, v_2) and span(w_1, w_2)? Is span(v_1, v_2) = span(w_1, w_2)?

$\{v_1, v_2\}$ is linearly independent

$\{w_1, w_2\}$ is linearly independent.

So they're planes in $\mathbb{R}^3$.

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row space of A = row space of rref (A).

$$\begin{pmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2}, & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ & 1, & -\frac{2}{3} \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & \frac{1}{3} \\ & 1, & -\frac{2}{3} \end{pmatrix}$$

the same,

$$\begin{pmatrix} 5 & -4 & 1 \\ 2 & 2 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & -9 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -\frac{2}{3} \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \end{pmatrix}$$

$\Rightarrow$ span(v_1, v_2) = span(w_1, w_2)

4. (10 points) For each matrix below, find rank(A). If A is invertible, find A^{-1}

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 4 & 6 & 9 \end{bmatrix}$

a). $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$

b) $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & & \\ 0 & 2 & 3 & & 1 & \\ 4 & 6 & 9 & & & 1 \end{array}\right] \xrightarrow[\text{reduce}]{\text{row}}$

$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & & \\ 0 & 2 & 3 & & 1 & \\ 0 & -2 & -3 & -4 & 0 & 1 \end{array}\right] \longrightarrow$

$\left[\begin{array}{ccc|c} 1 & 2 & 3 & \\ 0 & 2 & 3 & * \\ 0 & 0 & 0 & \end{array}\right]$ rank = 2

5. (10 points) The matrix $A = \begin{bmatrix} a & b & c \\ r & s & t \\ x & y & z \end{bmatrix}$ has determinant $\det(A) = 4$. Using properties of the determinant, find $\det(B)$ if $B = \begin{bmatrix} a-2r & b-2s & c-2\cancel{s} \\ 4r & 4s & 4t \\ -x+a & -y+b & -z+c \end{bmatrix}$ t.

$$\det B = 4 \begin{vmatrix} a-2r & b-2s & c-2 \\ r & s & t \\ -x+a & -y+b & -z+c \end{vmatrix}$$

$$= 4 \begin{vmatrix} a & b & c \\ r & s & t \\ -x & -y & -z \end{vmatrix}$$

$$= -4 \cdot 4 = -16$$

6. (10 points) Let $D = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $E = \begin{pmatrix} 1 & 2 & 0 & b_1 \\ 2 & 0 & 2 & b_2 \\ 2 & 1 & 1 & b_3 \end{pmatrix}$

(a) Compute the row reduced echelon form of E.

(b) Obtain the algebraic equation(s) that describe the vector space spanned by the column vectors of the matrix D in $\mathbb{R}^3$.

a).

$$\begin{bmatrix} 1 & 2 & 0 & b_1 \\ 2 & 0 & 2 & b_2 \\ 2 & 1 & 1 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & b_1 \\ 1 & 0 & 1 & \frac{b_2}{2} \\ 2 & 1 & 1 & b_3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 & \frac{b_2}{2} \\ 0 & 2 & -1 & b_1 - \frac{b_2}{2} \\ 0 & 1 & -1 & b_3 - b_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & \frac{b_2}{2} \\ 0 & 1 & 0 & b_1 - b_3 + \frac{b_2}{2} \\ 0 & 1 & -1 & b_3 - b_2 \end{bmatrix}.$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 & \frac{b_2}{2} \\ 0 & 1 & 0 & b_1 - b_3 + \frac{b_2}{2} \\ 0 & 0 & 1 & b_1 - 2b_3 + \frac{3b_2}{2} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & -b_2 - b_1 + 2b_3 \\ 0 & 1 & 0 & b_1 - b_3 + \frac{b_2}{2} \\ 0 & 0 & 1 & b_1 - 2b_3 + \frac{3}{2} b_2 \end{bmatrix}$$

b). The span of column vectors is $\mathbb{R}^3$, so no equations.

7. (10 points) Find the component vector of $x^2 + x + 1$ with respect to the basis $B = \{(x-1)(x-2), (x-2)(x-3), (x-3)(x-1)\}$.

$$f(x) = x^2 + x + 1 = a(x-1)(x-2) + b(x-2)(x-3) + c(x-3)(x-1)$$

$$f(1) = 2b = 3 \Rightarrow b = \frac{3}{2}$$

$$f(2) = -c = 7 \Rightarrow c = -7$$

$$f(3) = 2a = 13 \Rightarrow a = \frac{13}{2}$$

$$[f(x)]_B = \begin{pmatrix} \frac{13}{2} \\ \frac{3}{2} \\ -7 \end{pmatrix}$$

8. (10 points) Find a basis and the dimension of the subspace of $C^0(\mathbb{R})$ (space of continuous functions on the real line), given by

$$W = \text{span}\left\{e^{2x} + e^x, e^x, e^{2x}, x^2\right\}$$

$W = \text{span}\{e^x, e^{2x}, x^2\}$

$\{e^x, e^{2x}, x^2\}$ is linearly independent.

If $c_1 e^x + c_2 e^{2x} + c_3 x^2 = 0$.

$x=0$. $c_1 + c_2 = 0$.

$x=1$ $e c_1 + e^2 c_2 + c_3 = 0$

$x=-1$, $e^{-1} c_1 + e^{-2} c_2 + c_3 = 0$.

$$\det \begin{bmatrix} 1 & 1 & 0 \\ e & e^2 & 1 \\ e^{-1} & e^{-2} & 1 \end{bmatrix} = e^2 + e^{-1} - e^{-2} - e$$
$$= e^2 - e + e^{-1} - e^{-2} > 0.$$
$(e > 1)$

So $c_1 = c_2 = c_3 = 0$

Draft 1:

If you use this page and want it looked at, then you must indicate so on the page with the original problem on it. Make sure you label your work with the corresponding problem number.

Another method.

$x \to +\infty$, $\lim \frac{e^x}{e^{2x}} = 0$

$\lim \frac{x^2}{e^{2x}} = 0$

So $C_2 = 0$.

$C_1 e^x + C_3 x^2 = 0$, $C_1 + C_3 \frac{x^2}{e^x} = 0$.

$x \to +\infty$, $\lim \frac{x^2}{e^x} = 0$.

So $C_1 = 0$.

$C_3 x^2 = 0$. Choose $x = 1 \Rightarrow C_3 = 0$

Draft 2:

If you use this page and want it looked at, then you must indicate so on the page with the original problem on it. Make sure you label your work with the corresponding problem number.